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by Harry J. Koenig and Daniel I. Drain

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SUMMARY

The longitudinal dynamic characteristics of a linear structure may be approximated by a lumped-parameter model, which usually includes only the stiffness and mass distribution of the system. In general, the effect of damping is not included as a lumped parameter but is treated as an overall or modal characteristic superimposed on the results of the model. This report investigates the relation between the modal damping characteristics and the individual section dampers of a lumped-parameter system that could model a launch vehicle for longitudinal motion. The analysis presented shows that, for the typical damping range of launch vehicles, an approximately linear relation exists between local and modal damping. This linearity allows the determination of either local or modal damping from known values of the other as well as a method of correcting test stand data for the support system damping effects. Also, it can furnish insight into the relative effectiveness of each local damper on the modal damping at each resonant frequency. Two examples of the use of the analysis are included herein as well as a FORTRAN IV computer program suitable for calculations of the damping relations in lumped models up to 10 sections in size.

INTRODUCTION

A commonly used method for the dynamic analysis of structures is the development of a linear spring-mass model of the structure. The values of the masses and spring constants are derived directly from the weight and stiffness of the lumped section of the structure. The approximate resonant frequencies of the structure can then be found by solving for the resonant frequencies of the lumped parameter model. For a complete dynamic description of a structure, the damping characteristics must be known as well as the resonant frequencies. In the design stages or when the structure cannot be shaken experimentally, a modal damping is assumed through previous experience with similar structures.

Although the modal damping is a valuable entity, it is not sufficient to predict the effect of a modification to the structure. Through the use of individual section damping, the change in the structure dynamics because of a modification would be predicted more easily. A second valuable use would be in modeling the structure on the analog computer for longitudinal stability analysis.

This study investigated the relation between the individual section dampers and the modal damping of a series mass-spring system. The system considered would model a space vehicle structure for longitudinal motion. A method of solving for the damping characteristics of discrete systems has been proposed by DaDeppo (ref. 1) for the special case of damping proportional to either the mass or the spring matrix of the system. This report differs in that it is not limited to cases where special relations exist between the system constants, and it is written in terms of the physical coordinates of the system.

The relation of each individual section damper to the modal damping was found in terms of damping influence coefficients at each resonance for lightly damped (oscillatory) systems. Two examples of this method are given in the section APPLICATIONS. One example illustrates how to compensate for the influence of the suspension system damping on the experimental test data from a shaker facility.

ANALYSIS

The development of the relation between modal and section damping will be presented in steps. The equation of motion for a generalized single mass will be developed first; then these equations will be applied to the multimass system to form the system equations. Following this will be the procedure to solve the system equations, a brief discussion of modal modeling, and then the matching of the system equations to the modal equivalent model. This procedure yields equations that contain the information necessary to establish the general relation between modal and section damping. The general equations will then be applied to the physical case of a grounded two-mass system.

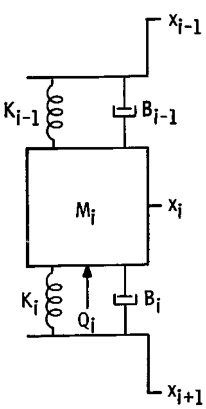


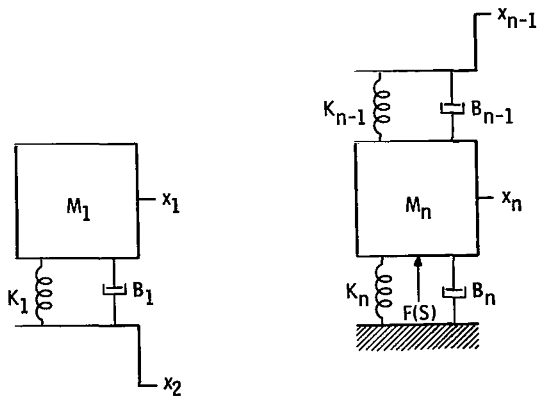
Figure 1. - Generalized coordinate system.

General Equations of Motion

The coordinate system and the related elements of mass, springs, and dampers for the i^{th} mass of a multimass system are illustrated in figure 1. The equation of motion about the i^{th} coordinate may be written immediately as

$$Q_i = M_i\ddot{x}_i + B_i(\dot{x}_i - \dot{x}_{i+1}) + K_i(x_i - x_{i+1})$$

$$- B_{i-1}(\dot{x}_{i-1} - \dot{x}_i) - K_{i-1}(x_{i-1} - x_i) \qquad (1)$$



(a) Top mass configuration. (b) Bottom mass configuration.

Figure 2. - Boundary conditions.

Collecting common coordinates and taking the Laplace transformation with zero initial conditions yield

$$\left[M_iS^2 + (B_{i-1} + B_i)S + K_{i-1} + K_i\right]x_i(S)$$

$$- (B_iS + K_i)x_{i+1}(S)$$

$$- (B_{i-1}S + K_{i-1})x_{i-1}(S) = Q_i(S) \qquad (2)$$

which is the desired form of the general equation about the i^{th} mass.

In addition to the general equation, two boundary conditions consistent with a series-type system were considered. The two conditions are the special constraint about the top and bottom mass of the system, as indicated in figure 2. The equation about the top mass may be derived from equation (2) by using the constraints $B_{i-1} = K_{i-1} = 0$; that is,

$$(M_1S^2 + B_1S + K_1)x_1(S) - (B_1S + K_1)x_2(S) = Q_1(S) \qquad (3)$$

The equation about the bottom mass may be obtained by using the constraint $x_{n+1} = 0$:

$$\left(M_nS^2 + (B_n + B_{n-1})S + K_n + K_{n-1}\right)x_n(S) - (B_{n-1}S + K_{n-1})x_{n-1}(S) = Q_n(S) \qquad (4)$$

Application of General Equation of Motion to n-Mass System

The equation of motion for an n-mass system, as illustrated in figure 3, was written in matrix form. For the problem considered, all generalized forces Q_i were set equal to zero except Q_n, the force on the bottom mass, where F(S) was substituted for $Q_n(S)$. The application of a sinusoidal force to the bottom mass is analogous to longitudinal shaking of a launch vehicle in a test facility. The equation of motion of the n-mass system in matrix form is

$$
\begin{bmatrix}
d_{11} & d_{12} & \cdot & \cdot & \cdot & d_{1i} & \cdot & \cdot & \cdot & d_{1n} \\
d_{21} & d_{22} & \cdot & \cdot & \cdot & d_{2i} & \cdot & \cdot & \cdot & d_{2n} \\
d_{31} & d_{32} & \cdot & \cdot & \cdot & d_{3i} & \cdot & \cdot & \cdot & d_{3n} \\
\cdot & \cdot & & & & \cdot & & & & \cdot \\
\cdot & \cdot & & & & \cdot & & & & \cdot \\
\cdot & \cdot & & & & \cdot & & & & \cdot \\
d_{ji} & d_{j2} & \cdot & \cdot & \cdot & d_{ji} & \cdot & \cdot & \cdot & d_{jn} \\
\cdot & \cdot & & & & \cdot & & & & \cdot \\
\cdot & \cdot & & & & \cdot & & & & \cdot \\
\cdot & \cdot & & & & \cdot & & & & \cdot \\
d_{n1} & d_{n2} & \cdot & \cdot & \cdot & d_{ni} & \cdot & \cdot & \cdot & d_{nn}
\end{bmatrix}
\begin{bmatrix}
x_i(S) \\ x_2(S) \\ x_5(S) \\ \cdot \\ \cdot \\ \cdot \\ x_i(S) \\ \cdot \\ \cdot \\ \cdot \\ x_n(S)
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F(S)
\end{bmatrix}
\tag{5}
$$

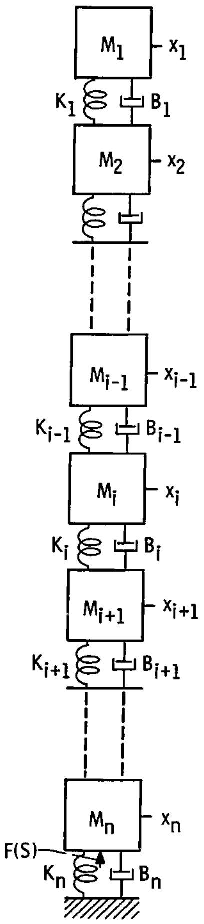


Figure 3. - Typical n-mass system.

where

$$d_{11} = M_1S^2 + B_1S + K_1$$

$$d_{ii} = M_iS^2 + (B_{i-1} + B_i)S + K_{i-1} + K_i$$

$$d_{nn} = M_nS^2 + (B_{n-1} + B_n)S + K_{n-1} + K_n$$

$$d_{i,\,i+1} = -(B_iS + K_i)$$

$$d_{i+1,\,i} = -(B_iS + K_i)$$

$$\text{all other } d_{i,\,j} = 0$$

From this matrix, the relation or transfer function between the force at the bottom mass and the motion of the top mass $x_1(S)/F(S)$ was selected for investigation because of the absence of antiresonances or node points. From equation (5), $x_1(S)/F(S)$ may be solved for by Cramer's rule (ref. 2), and the resulting solution will have the general form

$$\frac{x_1(S)}{F(S)} = \frac{(B_1S + K_1)(B_2S + K_2) \ldots (B_{n-1}S + K_{n-1})}{a_{2n}S^{2n} + a_{2n-1}S^{2n-1} + \ldots + a_1S + a_0} \tag{6}$$

In the general form of the transfer function, the coefficients in the numerator are the section dampers and spring constants. In the denominator, or characteristic polynomial, the coefficients $a_{2n}, a_{2n-1}, \ldots, a_0$ are functions of the springs, masses, and dampers. A method of evaluating these coefficients is provided in appendix D.

Modal Modeling

The classical approach to modal modeling is to solve for a set of independent modal solutions, the sum of which is equal to the transfer function of the system. In order to do this, a coordinate transformation must be made. This transformation is necessary to obtain the independence or orthogonality of the solutions; however, it limits the usefulness of the classical approach.

In order to avoid these limitations, the system transfer function was matched at each resonate peak by a second-order spring-mass-damper system, as illustrated in figure 4

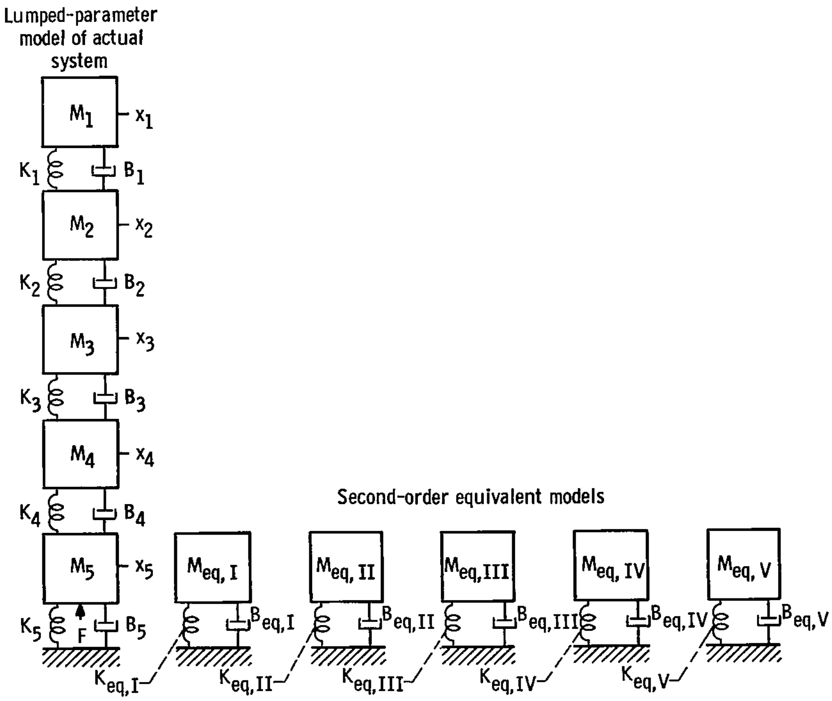


Figure 4. - Relation of lumped-parameter model to its modal equivalent models.

for a five-mass system. The mass, damper, and spring of each equivalent-system are represented by M_{eq}, B_{eq}, and K_{eq}, respectively; the additional Roman numeral subscript indicates the order of the mode. Thus, the modal transfer function evaluated at the first resonant mode ω_I would be

$$\left.\frac{x(S)}{F(S)}\right|_{\omega_I} = \frac{1}{M_{eq,I}S^2 + B_{eq,I}S + K_{eq,I}} \tag{7}$$

To complete the discussion of modal modeling, it is necessary to relate the modal constants with the frequency response characteristics of the system. This can be done readily when the damping ratio ξ of the system is on the order of 10 percent or less, and a plot of the amplitude as a function of frequency characteristics of the system has been obtained. Since the modeling is done at resonant condition, the resonant amplitude ratio and frequency ω_n are related to the modal equivalent values of mass, spring, and damper by the following equations:

$$\frac{B_{eq}}{M_{eq}} = 2\xi\omega_n \tag{8}$$

$$\frac{|x|_p}{|x|_{\omega=0}} \cong \frac{1}{2\xi} \tag{9}$$

$$\omega_n^2 = \frac{K_{eq}}{M_{eq}} \tag{10}$$

$$|x|_{\omega=0} = \frac{|F|}{K_{eq}} \tag{11}$$

Combining equations (8) to (11) yields

$$B_{eq} = \frac{|F|}{\omega_n |x|_p} \tag{12}$$

These are approximate equations for a second-order system; the assumption used to obtain these equations is $\xi < 0.1$ and, therefore, the peak, damped, and undamped frequencies are very nearly equal.

With the aforementioned definitions and equations, the relation between the section dampers and modal damping can be developed. The relation will be developed analytically first and then applied in two examples.

General Damping Equation

In the development of the final equation of this section (eq. (18)), three approximations will be used, the justifications of which will be discussed in the section Linearity Assumptions. The equation development is started by rewriting equation (6):

$$\frac{x_1(S)}{F(S)} = \frac{(B_1S + K_1)(B_2S + K_2)\ .\ .\ .\ (B_{n-1}S + K_{n-1})}{a_{2n}S^{2n} + a_{2n-1}S^{2n-1} + a_{2n-2}S^{2n-2} + \ .\ .\ .\ + a_1S + a_0} \tag{6}$$

Substituting $j\omega$ for S, collecting the real and imaginary terms, and taking the magnitude yield

$$\left|\frac{x_1(j\omega)}{F(j\omega)}\right| = \left|\frac{(jB_1\omega + K_1)(jB_2\omega + K_2)\ .\ .\ .\ (jB_{n-1}\omega + K_{n-1})}{\left[a_{2n}(-\omega^2)^n + a_{2n-2}(-\omega^2)^{n-1} + .\ .\ . + a_0\right] + j\left[a_{2n-1}(-\omega^2)^{n-1} + .\ .\ . + a_1\right]\omega}\right| \tag{13}$$

Evaluating at ω_Ω, which is the peak frequency of the Ω^{th} mode, permits the use of the approximation that the real portion of the denominator of equation (13) at a resonate peak is zero; that is,

$$a_{2n}\left(-\omega_\Omega^2\right)^n + a_{2n-2}\left(-\omega_\Omega^2\right)^{n-1} + \ .\ .\ .\ + a_2(-\omega^2) + a_0 = 0 \tag{14}$$

Substitution of equation (14) into equation (13) results in

$$\left|\frac{x_1(j\omega_\Omega)}{F(j\omega_\Omega)}\right| = \left|\frac{(jB_1\omega_\Omega + K_1)(jB_2\omega_\Omega + K_2)\ .\ .\ .\ (jB_{n-1}\omega_\Omega + K_{n-1})}{j\left[a_{2n-1}\left(-\omega_\Omega^2\right)^{n-1} + a_{2n-3}\left(-\omega_\Omega^2\right)^{n-2} + \ .\ .\ .\ + a_3\left(-\omega_\Omega^2\right) + a_1\right]\omega_\Omega}\right| \tag{15}$$

Assuming $B_i^2 << \left(K_i/\omega_\Omega\right)^2$ and inverting equation (15) yield

$$\frac{|F|}{|x|_p \omega_\Omega} = \frac{1}{\prod_{i=1}^{n-1} K_i} \left| \left[a_{2n-1} \left(-\omega_\Omega^2 \right)^{n-1} + a_{2n-3} \left(-\omega_\Omega^2 \right)^{n-2} + \ldots + a_1 \right] \right| \tag{16}$$

However, the left side of equation (16) is, from equation (12), equal to $B_{eq,\Omega}$. Equation (16) is, therefore, the general damping equation; that is,

$$B_{eq,\Omega} = \frac{1}{\prod_{i=1}^{n-1} K_i} \left| a_{2n-1} \left(-\omega_\Omega^2 \right)^{n-1} + a_{2n-3} \left(-\omega_\Omega^2 \right)^{n-2} + \ldots + a_1 \right| \tag{17}$$

In equation (17), the coefficients $a_{2n-1}, \ldots, a_1$ are linear functions of the local dampers B_i if it is assumed that in each a term the sum of the terms in B_i is much greater than the sum of the product terms (e.g., B_iB_j). With this assumption and a combination of like terms, equation (17) becomes a linear form that yields

$$B_{eq,\Omega} = \left| \alpha_1 B_1 + \alpha_2 B_2 + \ldots + \alpha_i B_i + \ldots + \alpha_n B_n \right| \tag{18}$$

where α_n is a combination of the systems masses, springs, and resonant frequencies.

Linearity Assumptions

In order to obtain the linear damping equations, three approximations are made. All approximations are consistent with light damping, as will be shown in this section. The first approximation, at the peak frequency ω_Ω, is

$$a_{2n} \left(-\omega_\Omega^2 \right)^n + a_{2n-2} \left(-\omega_\Omega^2 \right)^{n-1} + \ldots + a_2 \left(-\omega_\Omega^2 \right) + a_0 = 0 \tag{14}$$

The validity of this assumption can be argued from the limiting case of no damping. When there is no damping, the resonances are infinite and there will be no imaginary portion of the denominator in equation (13). Thus, the denominator consists only of equation (14), and it must go to zero in order to get an infinite resonance. The addition of a small amount of damping has two effects: the imaginary portion of the denominator in equation (13) exists and limits the resonant peak of the equation, and there is a slight modification in the a coefficients of equation (14) (appendix B), which will cause a small shift in the resonant frequency at which equation (14) goes to zero. This shift in resonant frequency is assumed to be negligibly small and is valid only for small values of damping.

The second approximation is

$$B_i^2 \ll \frac{K_i^2}{\omega_\Omega^2} \qquad i = 1, 2, 3, \ldots, n - 1 \tag{19}$$

The second approximation is used to eliminate the section damping from the numerator of equation (6). The effect of this assumption is to replace the product terms of $B_iS + K_i$ by a product of K_i's in the calculations of B_{eq} and M_{eq}. This relation generally holds for the lower resonant frequencies; it may not hold for the higher resonant frequencies, however, thus introducing errors into the solutions.

The third approximation is that each a coefficient in the imaginary polynomial (eq. (17)) is a linear function of the section dampers. This approximation is valid, as will be shown by example, if

$$\sum \text{Terms linear in } B_i \gg \sum \text{Product of } B_i \text{ terms} \tag{20}$$

where

$$i = 1, 2, 3, \ldots, n$$

This approximation was investigated for systems of up to seven masses. The terms that were linear in B outnumbered greatly the product of B terms. For light damping (i.e., small values of B), the products of the B terms are second- and third-order effects, which add to the validity of the approximation. As an example in a three-mass system, a product of B terms appears as a coefficient of S^3 (see appendix C). The total coefficient is

$$a_3 = (M_1K_2 + M_1K_3 + M_3K_2 + M_2K_2 + M_2K_3)B_1 + (M_1K_1 + M_1K_3 + M_2K_1 + M_2K_3)B_2$$
$$+ (M_1K_1 + M_1K_2 + M_2K_1)B_3 + B_1B_2B_3 \tag{21}$$

In this coefficient there are 12 terms that are linear in B (assuming that M and K are constant). There is one product of B terms, $B_1B_2B_3$. For light damping, the product of any mass spring MK is much larger than the product of damping coefficients BB.

The two other coefficients to be used in equation (17) for a three-mass system are

$$a_5 = M_1M_2B_2 + M_1M_2B_3 + M_1M_3B_1 + M_1M_3B_2 + M_2M_3B_1 \tag{22}$$

$$a_1 = K_2K_3B_1 + K_1K_2B_3 + K_1K_3B_2 \tag{23}$$

Coefficients a_5 and a_1 are linear in the section dampers without approximation (20). With the heuristic argument as given for coefficient a_3, the validity of the third approximation may be carried to an n-mass system.

Each of the approximations has been discussed in general to indicate how each will affect the linearity of equation (18). A specific example, however, shows the combined effects. In this example, the section dampers were all made equal, and the normalized B_{eq} for each mode was calculated for increasing values of section damping. The system model was the five-mass system used in the APPLICATIONS section of this report. The results are illustrated in figure 5 and indicate four distinct characteristics:

(1) The straight-line slope at low damping is identical to that predicted by the linearized equation (18).

(2) The first mode slope (rigid body) is a constant.

(3) The second and fourth mode slopes are lower than the initial slope for large values of damping.

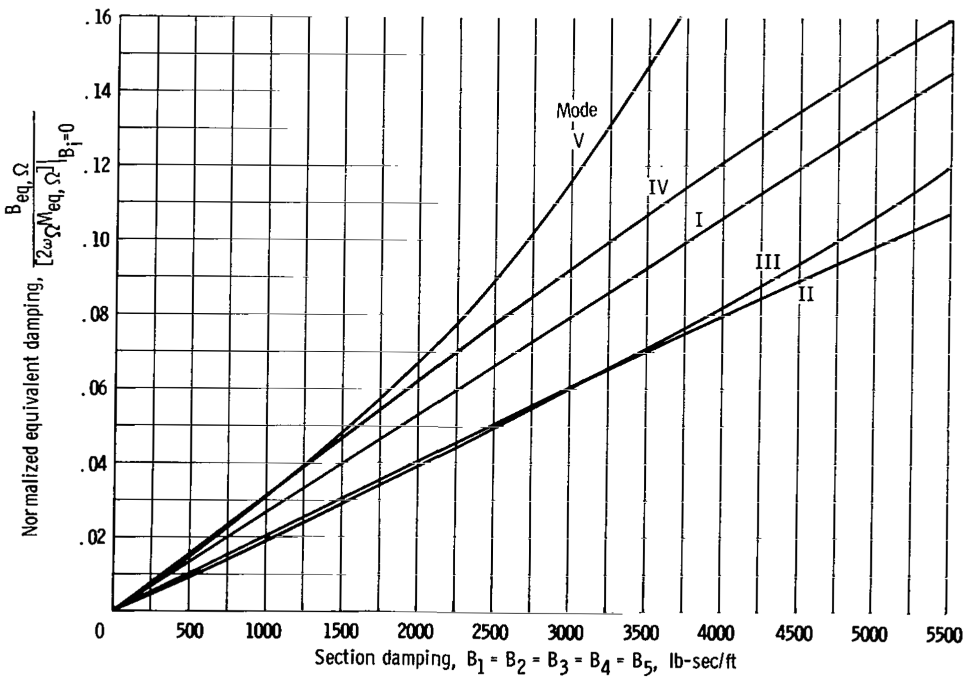


Figure 5. - Normalized equivalent damping as function of section damping.

(4) The third and fifth mode slopes rise above the initial slope for large damping.

The variances in slopes can be explained as follows: the rigid body mode behaves like a standard second-order system and, thus, should be linear; the sagging slope shows the influence of the violation of the second approximation; and the rising slope results from the resonant frequency shift due to damping. In this example, the frequency shift of the second and fourth modes is about 1 percent, while that for the third and fifth modes is 10 and 20 percent, respectively, at a damping value of 4000 pounds-second per foot. Thus, the two effects can be separated.

In figure 5, the normalized B_{eq} can be considered the critical damping factor of a second-order system. Thus, for this example, the linearized equation concept is reasonably accurate for all modes up to 5 percent modal damping and, except for the fifth mode, good to 10 percent of critical damping.

Damping Influence Coefficients

Around the resonances of a multiordered system, the modal damping can be approximated by a linear function of the section dampers (Linearity Assumptions). With the use of this linear function, the idea of damping influence coefficients may be introduced. The damping equation about the resonance Ω is

$$\left|\alpha_1 B_1 + \alpha_2 B_2 + \ldots + \alpha_i B_i + \ldots + \alpha_n B_n\right| = B_{eq,\Omega} \tag{18}$$

where α_i (i = 1, 2, . . ., n) are constants and called damping influence coefficients. Due to physical consideration, the α_i of equation (18) must be of the same sign or sustained oscillations could exist in a dissipative system. Therefore, α_i may be defined more explicitly as

$$\left|\alpha_i\right| = \frac{B_{eq,\Omega}}{B_i} \tag{24}$$

where

$$B_k = 0 \qquad k = 1, 2, 3, \ldots, n_i \qquad k \neq i$$

By using this process (i.e., setting all values of B equal to zero except the B of interest), the damping influence coefficients may be obtained with a digital computer. The programs used are given in appendix D. The influence coefficients are combinations of the systems masses, springs, and modal frequencies, as will be shown in the following sections.

Development of Damping Equations for Two-Mass System

For a two-mass system, the coefficients of the characteristic polynomial are found by taking the determinant of the system matrix of equation (5). The matrix for the two-mass system is

$$\begin{bmatrix} M_1S^2 + B_1S + K_1 & -(B_1S + K_1) \\ -(B_1S + K_1) & M_2S^2 + (B_1 + B_2)S + K_1 + K_2 \end{bmatrix} \tag{25}$$

In order to obtain the characteristic polynomial, the determinant of equation (25) is found. The polynomial generated is of the form of the denominator of equation (6), that is,

$$a_4S^4 + a_3S^3 + a_2S^2 + a_1S + a_0$$

where

$$a_4 = M_1M_2 \tag{26}$$

$$a_3 = M_1B_1 + M_1B_2 + M_2B_1 \tag{27}$$

$$a_2 = M_1K_1 + M_1K_2 + M_2K_1 + B_1B_2 \tag{28}$$

$$a_1 = K_2B_1 + K_1B_2 \tag{29}$$

$$a_0 = K_1K_2 \tag{30}$$

The linear damping equations for a two-mass system may be obtained from the general equation (eq. (17)):

$$B_{eq,I} = \frac{1}{K_1} \left| a_3\left(-\omega_I^2\right) + a_1 \right| \tag{31}$$

$$B_{eq,II} = \frac{1}{K_1} \left| a_3\left(-\omega_{II}^2\right) + a_1 \right| \tag{32}$$

When equations (27) and (29) are combined with equations (31) and (32), the following are obtained:

$$B_{eq,I} = \frac{1}{K_1} \left| \left[K_2 - (M_1 + M_2)\omega_I^2\right] B_1 + \left(K_1 - M_1\omega_I^2\right) B_2 \right| \tag{33}$$

$$B_{eq,II} = \frac{1}{K_1} \left| \left[K_2 - (M_1 + M_2)\omega_{II}^2\right] B_1 + \left(K_1 - M_1\omega_{II}^2\right) B_2 \right| \tag{34}$$

From equation (33), it can be seen that the damping influence coefficients for the first mode are

$$\alpha_1 = \frac{1}{K_1} \left| \left[K_2 - (M_1 + M_2)\omega_I^2\right] \right| \tag{35}$$

$$\alpha_2 = \frac{1}{K_1} \left| \left(K_1 - M_1\omega_I^2\right) \right| \tag{36}$$

The physical significance of the influence coefficient is that α_i is the amount of modal equivalent damping contributed by B_i when B_i has a unit value of damping (i.e., $B = 1$ lb-sec/ft).

APPLICATIONS

After the linear damping equations and modal dampings are found, a system of n simultaneous nonhomogeneous linear equations is obtained. These equations have the section dampers as their variables and may be solved simultaneously.

The solution to the system of equations is the actual value of section dampers, which, if substituted into the spring-mass model, would give a reproduction of the amplitude frequency plot. A Gaussian reduction program is supplied in appendix D for the solution of the simultaneous equations. Two examples of the general technique are given in the following sections.

Example 1

If the spring-mass system (fig. 6) with the parameter values of

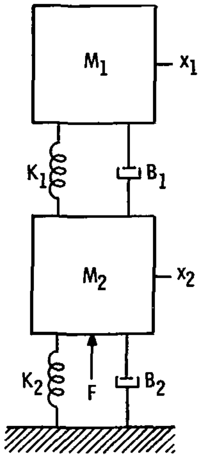


Figure 6. - Two-mass system.

$$M_1 = 1 \qquad M_2 = 2$$

$$K_1 = 20 \qquad K_2 = 10$$

$$B_1 = 0.1 \qquad B_2 = 0.05$$

is given, the plot of amplitude X_1 as a function of frequency may be calculated directly from the theoretical transfer function of the system. This amplitude-frequency plot is presented in figure 7. For this example, it could be assumed that the plot is data and B_1 and B_2 of the aforementioned model are unknown. The problem is finding the section dampers, given the spring-mass model and the amplitude frequency plot.

The two linear damping equations for the fourth-order system are

$$B_{eq,I} = \frac{1}{K_1}\left|\left[K_2 - (M_1 + M_2)\omega_I^2\right]B_1 + \left(K_1 - M_1\omega_I^2\right)B_2\right| \tag{33}$$

$$B_{eq,II} = \frac{1}{K_1}\left|\left[K_2 - (M_1 + M_2)\omega_{II}^2\right]B_1 + \left(K_1 - M_1\omega_{II}^2\right)B_2\right| \tag{34}$$

Four values must now be obtained from the experimental amplitude-frequency plot (fig. 7). These values are the peak amplitudes and resonant frequencies:

$$|x|_{p,I} = 12.5 \qquad \omega_I = 1.77$$

$$|x|_{p,II} = 0.385 \qquad \omega_{II} = 5.64$$

Substituting the values of K_1, K_2, M_1, M_2, ω_I, and ω_{II} into equations (33) and (34) results in

$$B_{eq,I} = \left|0.0292B_1 + 0.8430B_2\right| \tag{37}$$

$$B_{eq,II} = \left|-4.279B_1 - 0.5930B_2\right| \tag{38}$$

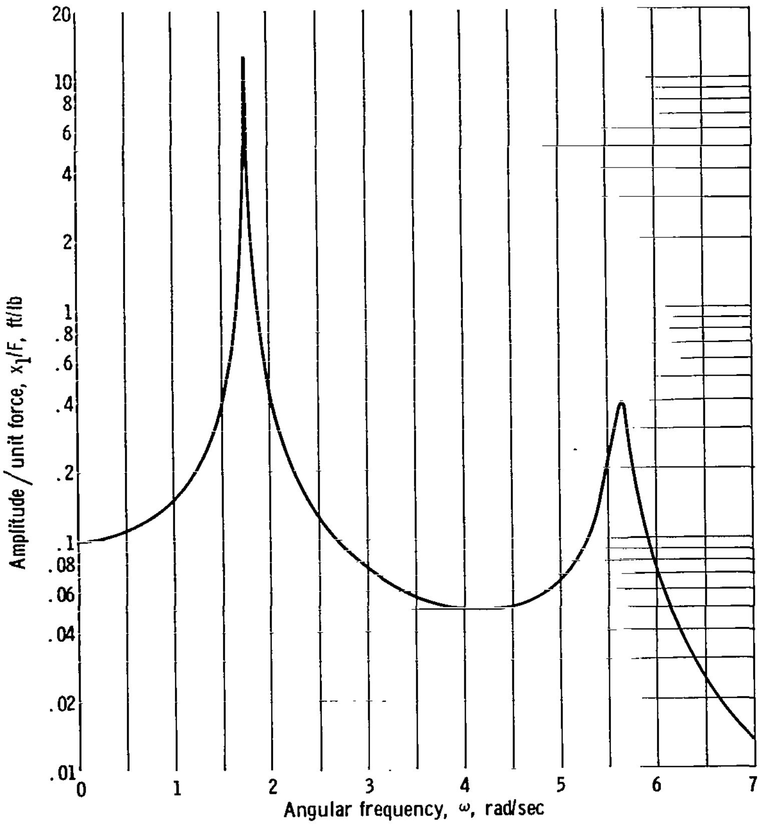


Figure 7. - Amplitude-frequency plot for fourth-order system. Parameters: mass, $M_1 = 1$ and $M_2 = 2$; spring constant, $K_1 = 20$ and $K_2 = 10$; damping, $B_1 = 0.1$ and $B_2 = 0.05$.

From equation (12),

$$B_{eq, I} = \frac{1}{(1.77)(12.5)} = 0.0451 \tag{39}$$

$$B_{eq, II} = \frac{1}{(5.64)(0.385)} = 0.461 \tag{40}$$

Solving equations (37) and (38) yields

$$B_1 = 0.102 \qquad B_2 = 0.050$$

which are the values of section dampers that would generate figure 7 with the given spring-mass model.

As the number of sections increases, the problem becomes more difficult, and solution by hand, as in example 1, must be abandoned. Example 2 has five sections, and the problem is solved by using the digital computer.

Example 2

Part A. - A problem that might be presented in practice would be the evaluation of the section dampers of a launch vehicle if the following conditions had been filled:

(1) The vehicle had been shaken in a test stand.

(2) The plot of longitudinal amplitude as a function of frequency had been obtained (fig. 8).

(3) The spring-mass model had been obtained from the weight and spring distribution of the vehicle.

TABLE I. - ASSUMED FIVE-MASS SYSTEM PARAMETERS

Section	Mass, slug	Spring, lb/ft	Damper, lb-sec/ft
1	1060	8.65×10^6	500
2	68	5.35	900
3	58	14.0	640
4	98	18.0	1300
5	178	.21	1600

In order to demonstrate the technique to be used in the solution to this problem, figure 8 was calculated from the transfer function of a five-mass system with the parameters given in table I (see fig. 4, p. 5) and was assumed to be the experimental data. Presuming that the experimental data in figure 8 and the spring-mass model are available, the problem is to find the dampers in the vehicle that are characteristic to the generated plot.

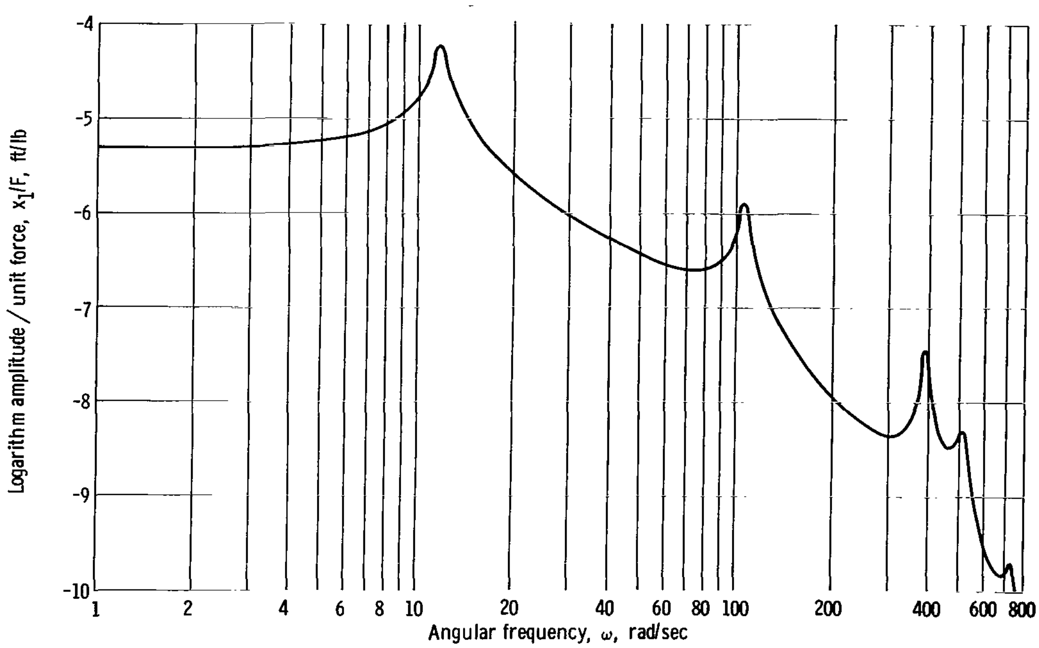


Figure 8. - Tenth-order amplitude-frequency plot with assumed dampers.

The main program (appendix D) is used, and the lumped-mass and spring constants are applied to develop the following linear damping equations:

$$\left.\begin{aligned}
B_{eq,I} &= 0.00031B_1 + 0.00089B_2 + 0.00016B_3 + 0.00011B_4 + 0.93354B_5 \\
B_{eq,II} &= 0.52103B_1 + 1.2987B_2 + 0.13964B_3 + 0.03485B_4 + 3.5894B_5 \\
B_{eq,III} &= 47.788B_1 + 2.1330B_2 + 14.120B_3 + 16.146B_4 + 7.2588B_5 \\
B_{eq,IV} &= 93.375B_1 + 256.54B_2 + 1.4449B_3 + 77.775B_4 + 11.395B_5 \\
B_{eq,V} &= 87.102B_1 + 2389.2B_2 + 5087.6B_3 + 1507.7B_4 + 51.669B_5
\end{aligned}\right\} \tag{41}$$

Inversion of the matrices of equation (41) results in

$$\left.\begin{aligned}
B_1 &= -2.732B_{eq,I} + 0.6818B_{eq,II} + 0.01937B_{eq,III} - 0.00293B_{eq,IV} - 0.715\times10^{-4}B_{eq,V} \\
B_2 &= -1.923B_{eq,I} + 0.5123B_{eq,II} - 0.00813B_{eq,III} + 0.001302B_{eq,IV} + 0.819\times10^{-5}B_{eq,V} \\
B_3 &= -1.876B_{eq,I} + 0.4938B_{eq,II} + 0.002445B_{eq,III} - 0.00416B_{eq,IV} + 1.77\times10^{-4}B_{eq,V} \\
B_4 &= 9.502B_{eq,I} - 2.5176B_{eq,II} + 0.003516B_{eq,III} + 0.01216B_{eq,IV} + 0.556\times10^{-4}B_{eq,V} \\
B_5 &= 1.073B_{eq,I} - 0.000501B_{eq,II} + 0.4\times10^{-6}B_{eq,III} - 0.1\times10^{-5}B_{eq,IV} - 0.191\times10^{-7}B_{eq,V}
\end{aligned}\right\} \tag{42}$$

The experimental values of B_{eq} must be found now from the experimental amplitude - frequency plot (fig. 8). From this plot, the resonant amplitudes and frequencies were obtained and are listed in table II along with the values of $B_{eq,\Omega}$ calculated from equation (12).

The equivalent dampings are then substituted into the inverted matrix, and the section dampers are calculated. In table III, the calculated damping values are compared with the assumed damping values of table I.

A perfect agreement between the calculated and assumed section damper values should not be expected, since there were errors in reading the values of B_{eq} from

TABLE II. - SYSTEM MODAL CHARACTERISTICS

BASED ON FIGURE 8

Quadratic	Resonant frequency, ω	Amplitude	Second-order modal system damping, B_{eq}
I	11.6	$10^{-4.24}$	1495
II	105	$10^{-5.89}$	7390
III	390	$10^{-7.42}$	67500
IV	512	$10^{-8.29}$	381000
V	734	$10^{-9.71}$	6990000

TABLE III. - COMPARISON OF

CALCULATED AND ASSUMED

DAMPING VALUES

Section damper	Calculated damping using linear damping equations	Assumed section damper used for figure 8
B_1	641	500
B_2	916	900
B_3	661	640
B_4	860	1300
B_5	1600	1600

TABLE IV. - EVALUATION OF CALCULATION AND READING ERRORS

Case	Error in reading second-order equivalent mode damping, percent					Resulting error in damping, percent				
	$B_{eq,I}$	$B_{eq,II}$	$B_{eq,III}$	$B_{eq,IV}$	$B_{eq,V}$	B_1	B_2	B_3	B_4	B_5
1	0	0	0	0	0	8.4	2.2	4.4	6.7	0
2	.1	1.2	-.3	-2.2	-2.4	28.2	1.8	3.2	33.8	0

figure 8 and since the analysis uses approximations. Two cases of the expected errors for this example are cited in table IV. Case 1 indicates the analysis approximation errors, and case 2 is the data used in this example. The large errors in the values of B_1 and B_4 for the second case are not as important as might be expected. In fact, they indicate the lack of influence of these dampers on the transfer function. This lack of influence can be demonstrated by calculating the transfer function with the results of this second example and comparing it with the original function. When this is done, the curve shapes are identical. The amplitude of the peaks for modes I to V are in error by 0.0, 1.1, 0.1, 4.2, and 6.7 percent, which is not much more in error than the original reading errors of the example.

The masses and spring constants of example 2 are typical of an actual two-stage vehicle with conditions close to first-stage cutoff and supported in a shake stand. The symbols B_5 and K_5 represent the shake-stand suspension system damping and spring constant. Inspection of equation (41), which relates the modal to section damping, will show that the stand damping B_5 constitutes 100 percent of the rigid body-mode damping $B_{eq,I}$, as it should, but it also contributes 80 percent of the measured first-mode damping $B_{eq,II}$, which is undesirable. A reduction in the magnitude of the stand damping from $B_5 = 1600$ to $B_5 = 160$ will reduce the stand damping contribution to the first mode to 25 percent, which is still substantial and which readily points out the need for compensating for the suspension system damping as the damping characteristics are being evaluated. Node point suspension, of course, is the ideal way to eliminate the effect of stand damping, but this, in general, is not practical for longitudinal dynamic tests. A method to eliminate the stand damping based on the technique presented herein is given in the next section.

Part B. - By simply subtracting the stand damping, the spring-mass model may be used to extrapolate to the value of damping that would exist in a vehicle in free flight. The first step is to perform an analysis, as in Part A, which uses the B_{eq} found from the amplitude-frequency plot. Equation (43) may then be written as

$$\left.\begin{aligned} B_{eq,II} - 3.5894B_5 &= 0.52103B_1 + 1.2987B_2 + 0.13964B_3 + 0.03485B_4 \\ B_{eq,III} - 7.3588B_5 &= 47.788B_1 + 2.1330B_2 + 14.12B_3 + 16.146B_4 \\ B_{eq,IV} - 11.395B_5 &= 93.375B_1 + 256.54B_2 + 1.4449B_3 + 77.775B_4 \\ B_{eq,V} - 51.669B_5 &= 87.102B_1 + 2389.2B_2 + 5087.6B_3 + 1507.7B_4 \end{aligned}\right\} \quad (43)$$

The linear damping equations, however, may be obtained for a free-free system as well as a grounded system. The method is the same as that used in appendix B, but the bottom spring and damper are set at zero. If the damping equations of the free-free system in Part A are generated, the linear damping equations obtained are

$$\left.\begin{aligned} B_{eq,II,ff} &= 0.4963B_1 + 1.248B_2 + 0.13748B_3 + 0.03731B_4 \\ B_{eq,III,ff} &= 47.811B_1 + 2.069B_2 + 14.124B_3 + 16.2808B_4 \\ B_{eq,IV,ff} &= 93.908B_1 + 257.31B_2 + 1.4069B_3 + 77.819B_4 \\ B_{eq,V,ff} &= 87.349B_1 + 2395.5B_2 + 5099.4B_3 + 1509.3B4 \end{aligned}\right\} \quad (44)$$

The similarity between equations (43) and (44) leads to the approximations

$$
\left.\begin{aligned}
B_{eq,II,ff} &\cong B_{eq,II} - 3.5894B_5 \\
B_{eq,III,ff} &\cong B_{eq,III} - 7.3588B_5 \\
B_{eq,IV,ff} &\cong B_{eq,IV} - 11.395B_5 \\
B_{eq,V,ff} &= B_{eq,V} - 51.669B_5
\end{aligned}\right\} \tag{45}
$$

These equations illustrate that, even though the vehicle section dampers were not found with accuracy, the model of the vehicle will give the amplitude-frequency plot of the vehicle in free flight if the stand damping B_5 is known. This is not an extremely stringent condition since B_5 may be determined from the rigid body mode $B_{eq,I}$ (eq. (42)).

The influence coefficients can also be used as a guide to the most effective location for section damping of a vehicle during free flight. Thus, for the first free-free mode (eq. (44)), it can be seen that, by increasing B_2, the overall damping of that mode is increased optimally by 1.248 times the damping introduced at B_2. If damping could be added to any section with equal facility (not true in practice), the optimum placement to increase overall effective damping would be at section 2.

CONCLUSIONS

For longitudinal oscillations of lightly damped structures, which are usually characteristic of launch vehicles, the relation of the overall modal damping to section damping has been shown to be approximately a linear function. A general procedure is furnished that will develop the influence coefficient of each section damper on the system damping. These influence coefficients thus can be used in an explanation of the effect of changes in local damping magnitude on the overall vehicle damping. They can also be used to provide insight into how the test vehicle suspension system damping will affect the measurement of vehicle damping characteristics in a test stand.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, June 21, 1966,
180-31-01-01-22.

APPENDIX A

SYMBOLS

[A] matrix of order 2n

a characteristic polynomial coefficient (see eq. (6))

B damping, lb-sec/ft

[C] manipulative matrix (see appendix D)

d matrix component (eq. (5))

F force, lb

[I] identity matrix

j $\sqrt{-1}$

K spring constant, lb/ft

M mass, slugs

n number of masses in system

Q generalized force, lb

S Laplace operator, $\sec^{-1}$

x generalized coordinate, ft, $n \times 1$ column vector

α damping influence coefficient, dimensionless

γ defined by eq. (D8)

μ number of resonances in system

ξ damping ratio, dimensionless

ρ defined by eq. (D7)

φ_1 first state variable defined as x, $(n \times 1)$ column vector

φ_2 second state variable defined as dx/dt, $(n \times 1)$ column vector

$[\psi]$ state vector, $(2n \times 1)$ column vector

ω angular frequency, rad/sec

ω_n resonant angular frequency, rad/sec

Subscripts:

eq equivalent second-order (modal) system value

ff free-free system

i index of lumped-parameter system 1, 2, 3, . . ., n

k dummy indexing variable

ℓ index for γ and exponent of [A]

p peak

Ω index of quadratic term (eq. (20)) I, II, III, . . ., μ

APPENDIX B

DERIVATION OF MODAL EQUIVALENT MASS

Grouping the odd and even powers of S from equation (6) yields

$$\frac{x_1(S)}{F(S)} = \frac{(B_1S + K_1)(B_2S + K_2) \ . \ . \ . \ (B_{n-1}S + K_{n-1})}{\left(a_{2n}S^{2n} + a_{2n-2}S^{2n-2} + . \ . \ . + a_0\right) + \left(a_{2n-1}S^{2n-2} + a_{2n-3}S^{2n-4} + . \ . \ . + a_1\right)S} \tag{B1}$$

The term

$$a_{2n}S^{2n} + a_{2n-2}S^{2n-2} + a_{2n-4}S^{2n-4} + . \ . \ . + a_2S^2 + a_0 \tag{B2}$$

may be factored into the form

$$a_{2n}\left(S^2 + \omega_I^2\right)\left(S^2 + \omega_{II}^2\right)\left(S^2 + \omega_{III}^2\right) \ . \ . \ . \ \left(S^2 + \omega_\mu^2\right) \tag{B3}$$

where ω_I, ω_{II}, ω_{III}, . . ., ω_μ are the frequencies where the real polynomial (eq. (14)) goes to zero. This may be shown readily when $j\omega$ is substituted for S. With equation (B3), equation (B1) may be rewritten as

$$\frac{x_1(S)}{F(S)} = \frac{(B_1S + K_1)(B_2S + K_2) \ . \ . \ . \ (B_{n-1}S + K_{n-1})}{a_{2n}\left(S^2 + \omega_{II}^2\right)\left(S^2 + \omega_{III}^2\right) \ . \ . \ . \ \left(S^2 + \omega_\mu^2\right)S^2 + \left(a_{2n-1}S^{2n-1} + . \ . \ . + a_1\right)S + a_{2n}\left(S^2 + \omega_{II}^2\right) \ . \ . \ . \ \left(S^2 + \omega_\mu^2\right)\omega_I^2} \tag{B4}$$

Equation (B4) is then conformed to equation (7):

$$M_{eq,I} = \left|\frac{a_{2n}\left(S^2 + \omega_{II}^2\right)\left(S^2 + \omega_{III}^2\right) . \ . \ . \ \left(S^2 + \omega_\mu^2\right)}{(B_1S + K_1)(B_2S + K_2)(B_3S + K_3) \ . \ . \ . \ (B_{n-1}S + K_{n-1})}\right|_{S=j\omega_I} \tag{B5}$$

$$B_{eq,I} = \left|\frac{a_{2n-1}S^{2n-2} + a_{2n-3}S^{2n-4} + . \ . \ . + a_1}{(B_1S + K_1)(B_2S + K_2) \ . \ . \ . \ (B_{n-1}S + K_{n-1})}\right|_{S=j\omega_I} \tag{B6}$$

$$K_{eq,I} = M_{eq,I}\omega_I^2 \tag{B7}$$

APPENDIX C

COEFFICIENTS OF CHARACTERISTIC EQUATION OF THREE-MASS GROUNDED SYSTEM

The coefficients of the characteristic equation of a three-mass grounded system are as follows:

$$a_6 = M_1M_2M_3 \tag{C1}$$

$$a_5 = M_1M_2B_2 + M_1M_2B_3 + M_1M_3B_1 + M_1M_3B_2 + M_2M_3B_1 \tag{C2}$$

$$a_4 = M_1M_2K_2 + M_1M_2K_3 + M_1M_3K_1 + M_1M_3K_2 + M_1B_1B_2 + M_1B_1B_3 + M_1B_2B_3 + B_1B_2M_2 + B_1B_2M_3 + B_1B_3M_2 + K_1M_2M_3 \tag{C3}$$

$$a_3 = M_1K_1B_2 + M_1K_3B_2 + M_1K_2B_1 + M_1K_3B_1 + M_1K_1B_3 + M_1K_2B_3 + M_3K_2B_1 + M_2K_2B_1 + M_2K_3B_1 + B_1B_2B_3 + M_3K_1B_2 + M_2K_1B_2 + M_2K_1B_3 \tag{C4}$$

$$a_2 = M_1K_1K_2 + M_1K_1K_3 + M_1K_2K_3 + K_2B_1B_3 + K_3B_1B_2 + M_3K_1K_2 + M_2K_1K_2 + M_2K_1K_3 + K_1B_2B_3 \tag{C5}$$

$$a_1 = K_2K_3B_1 + K_1K_2B_3 + K_1K_3B_2 \tag{C6}$$

$$a_0 = K_1K_2K_3 \tag{C7}$$

APPENDIX D

DESCRIPTION AND LISTING OF DAMPER

The main FORTRAN IV program and flow chart used in calculating the influence coefficients are presented herein. The main program, DAMPER, has three subroutines called COEFF, GAUSSR, and POLYRT. Subroutines COEFF, which finds the coefficients of the characteristic polynomial, and GAUSSR, a Gaussian reduction subroutine, are presented herein. The subroutine POLYRT, which is a common polynomial root finding subroutine, is not supplied.

The technique used in generating the influence coefficients is to set all dampers of the system to zero except the one in question, as is stated in the text in the sections General Damping Equation and Damping Influence Coefficients; the corresponding B_{eq} is then calculated.

The flow chart for DAMPER is given in figure 9. The FORTRAN IV program for DAMPER is presented after figure 9.

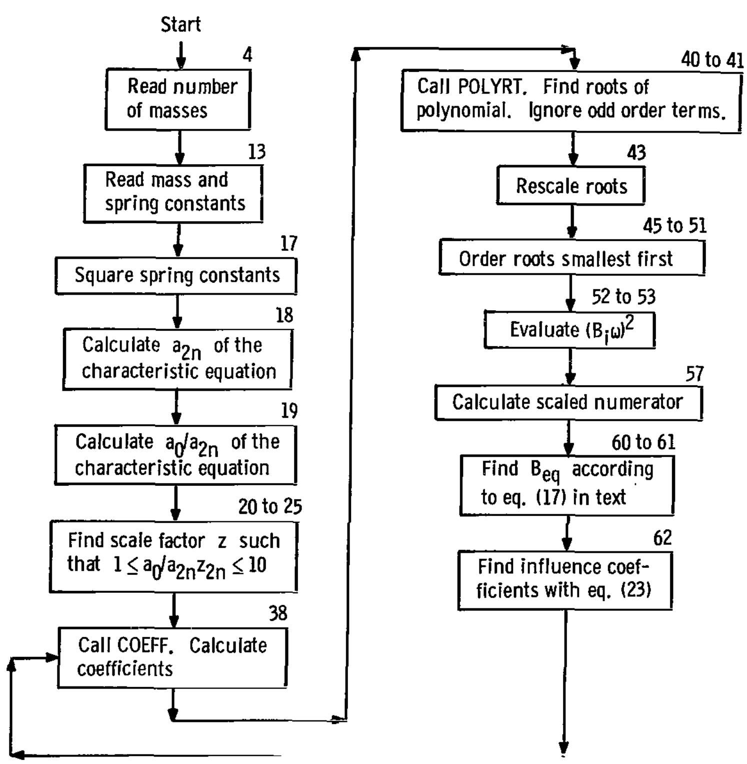


Figure 9. - Main program-DAMPER.

```
          DIMENSION OO(20),V(11),BB(10,11),B(11),R(40),P(40),AA(40)
          DIMENSION SZ(11)
          COMMON/VALUES/S(11),W(11),BV(11),SS(21),Z,NUM
          COMMON/QUANIT/ZETA(11),C(10,11),ZL
          EQUIVALENCE (B,BV),(BB,C)
C         NUMBER OF MASSES (N) = NUM
C         SCALE FACTOR = Z
C         MASS CONSTANT (M) = W
C         DAMPING CONSTANT (B) = B
C         SPRING CONSTANT (K) = S
        1 FORMAT(I2)
        2 FORMAT(2X,10F12.6)
        3 FORMAT(2X,2F20.8)
        4 READ(5,1)(NUM)
        5 ZL=NUM
        6 LA=2*NUM
        7 MO=2*LA
        8 MA=NUM-1
        9 DO 11 I=1,11
       10 S(I)=1.0
       11 W(I)=1.0
       12 DO 13 LUD=1,NUM
       13 READ(5,3)(W(LUD),S(LUD))
       14 WRITE(6,3)(W,S)
       15 DO 17 I=1,MA
       16 IJK=I+NUM
       17 OO(IJK)=S(I)**2
       18 ZW=W(1)*W(2)*W(3)*W(4)*W(5)*W(6)*W(7)*W(8)*W(9)*W(10)
       19 ZS=S(1)*S(2)*S(3)*S(4)*S(5)*S(6)/ZW*S(7)*S(8)*S(9)*S(10)
       20 DO 23 I=1,100
       21 ZS=ZS/10.
       22 IF(ZS-10.)24,24,23
       23 CONTINUE
       24 PL=I
       25 Z=-PL/ZL
       26 WRITE(6,3)(ZW,PL,ZL,Z,ZS)
       27 DO 29 I=1,10
       28 V(I)=1.0
       29 ZETA(I)=1.0
       30 DO 32 IA=1,10
       31 DO 32 JA=1,10
       32 BB(IA,JA)=0.0
       33 DO 63 IVY=1,NUM
       34 DO 62 IZZ=1,NUM
       35 DO 36 IW=1,10
       36 B(IW)=0.0
       37 B(IZZ)=ZETA(IZZ)
       38 CALL COEFF
       39 DO 40 LP=1,11
       40 SZ(LP)=SS(2*LP-1)
       41 CALL POLYRT(SZ,R,NUM,1)
       42 DO 43 IV=1,LA
       43 R(IV)=SQRT(ABS(R(IV)))*10.**(-Z/2.)
       44 MX=2*MA
       45 DO 51 K=2,MX,2
       46 DO 51 J=K,MX,2
       47 IF(ABS(R(K))-ABS(R(J+2)))51,51,48
       48 P(K)=R(J+2)
       49 R(J+2)=R(K)
       50 R(K)=P(K)
       51 CONTINUE
       52 DO 53 LL=1,MA
       53 OO(LL)=(B(LL)*R(2*IVY))**2
       54 OY=1.
       55 DO 57 I=1,MA
       56 IDZ=I+NUM
       57 OY=SQRT(OO(I)+OO(IDZ))*OY*10.**((2.*ZL-1.)*Z/(2.*(ZL-1.)))
       58 OM=R(2*IVY)*10.**(Z/2.)
       59 WO=OM**2
       60 BEQ=(1./OY)*((((SS(10)*WO-SS(8))*WO+SS(6))*WO-SS(4))*WO+SS(2))
       61 BEQ=ZW*ABS(BEQ)
       62 BB(IVY,IZZ)=BEQ/B(IZZ)
       63 CONTINUE
       64 WRITE(6,2)(BB)
       65 CALL GAUSSR
       66 STOP
       67 END
```

DEVELOPMENT OF METHOD TO OBTAIN COEFFICIENTS OF CHARACTERISTIC POLYNOMIAL

The method used is a modification of Leverriers method, as found in reference 3. Equation (5) may be rewritten in matrix-vector form such that

$$\left[\left[\diagdown M \diagdown\right]S^2 + [B]S + [K]\right]\left[x(S)\right] = \left[F(S)\right] \tag{D1}$$

Then, to obtain the characteristic equation, [F(S)] is set equal to zero, and equation (D1) is reduced with the substitutions

$$\varphi_1(S) = x(S) \tag{D2}$$

$$\dot{x}(S) = S\varphi_1(S) = \varphi_2(S) \tag{D3}$$

to the form

$$\left[\dot{\psi}(S)\right] = [A]\left[\psi(S)\right] \tag{D4}$$

where

$$\left[\psi(S)\right] = \begin{bmatrix} \varphi_1(S) \\ \varphi_2(S) \end{bmatrix} \tag{D5}$$

$$[A] = \begin{bmatrix} [O] & [I] \\ -[K][M]^{-1} & -[B][M]^{-1} \end{bmatrix} \tag{D6}$$

The coefficients are then found as a function of the trace of the powers of the [A] matrix. Symbolically, the general term is

$$a_{2n-k} = \rho_k = -\frac{1}{k}\left(\rho_{k-1}\gamma_1 + \rho_{k-2}\gamma_2 + \ldots + \rho_1\gamma_{k-1} + \gamma_k\right) \qquad k = 1,\ 2,\ \ldots,\ 2n \tag{D7}$$

$$\gamma_\ell = \text{trace}\ \ [A]^\ell \tag{D8}$$

Then

$$a_{2n-1} = -\gamma_1 \tag{D9}$$

$$a_{2n-2} = -\frac{1}{2}(\rho_1\gamma_1 + \gamma_2) \tag{D10}$$

$$a_{2n-3} = -\frac{1}{3}(\rho_2\gamma_1 + \rho_1\gamma_2 + \gamma_3) \tag{D11}$$

$$a_{2n-k} = \rho_k \tag{D12}$$

$$a_{2n} = 1 \tag{D13}$$

The 2n+1 coefficients of the characteristic polynomial may be generated quite rapidly by this repetitive procedure.

The flow charts for COEFF and GAUSSR are given in figures 10 and 11, respectively. The subroutines are presented after figures 10 and 11.

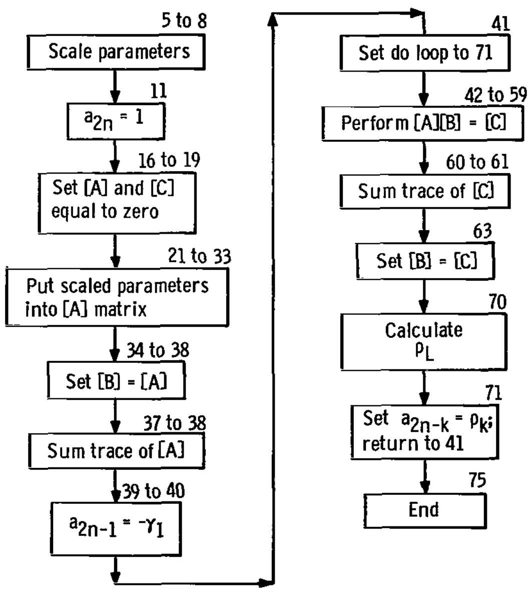


Figure 10. - Flow chart for subroutine COEFF.

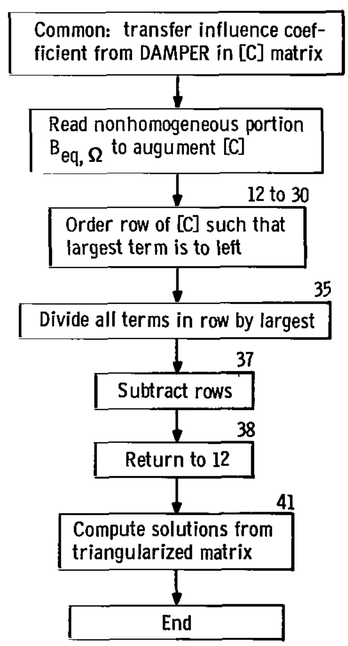


Figure 11. - Flow chart for subroutine GAUSSR.

```
      SUBROUTINE COEFF
      DOUBLE PRECISION A,B,C,SIGMA,CC,CS,P,BZ,SZ,WZ
      DIMENSION A(20,20),B(20,20),C(20,20),CS(21),CC(20),SIGMA(20),
     1D(11),BZ(11),SZ(11),WZ(11)
      COMMON/VALUES/S(11),W(11),BV(11),SS(21),Z,NUM
C     DAMPING CONSTANT (B) = BV
C     SCALED MASS CONSTANT = WZ
C     SCALED DAMPING CONSTANT = BZ
C     SCALED SPRING CONSTANT =SZ
C     RHO(L)=CC(L)
    1 LU=NUM-1
    2 LY=NUM
    3 LW=2*LY
    4 LX=NUM+1
    5 DO 8 I=1,LY
    6 BZ(I+1)=(10.**(Z/2.))*BV(I)
    7 SZ(I+1)=(10.**Z)*S(I)
    8 WZ(I+1)=W(I)
    9 BZ(1)=0.0D0
   10 SZ(1)=0.0D0
   11 SS(LW+1)=1.0
   12 CS(1)=1.0
   13 DO 15 L=1,LW
   14 CC(L)=0.0D0
   15 SIGMA(L)=0.0D0
   16 DO 19 I=1,LW
   17 DO 19 J=1,LW
   18 A(I,J)=0.0D0
   19 C(I,J)=0.0D0
   20 CONTINUE
   21 DO 26 I=1,LY
   22 LV=I+LY
   23 A(I,LV)=1.0
   24 A(LV,LV)=-(BZ(I)+BZ(I+1))/WZ(I+1)
   25 A(LV,I)=-(SZ(I)+SZ(I+1))/WZ(I+1)
   26 CONTINUE
   27 DO 33 I=1,LU
   28 LV=I+LY
   29 A(LV+1,LV)=BZ(I+1)/WZ(I+1)
   30 A(LV,LV+1)=BZ(I+1)/WZ(I+2)
   31 A(LV,I+1)=SZ(I+1)/WZ(I+2)
   32 A(LV+1,I)=SZ(I+1)/WZ(I+1)
   33 CONTINUE
   34 DO 36 I=1,LW
   35 DO 36 J=1,LW
   36 B(I,J)=A(I,J)
   37 DO 38 I=1,LW
   38 SIGMA(1)=A(I,I)+SIGMA(1)
   39 CC(1)=-SIGMA(1)
   40 SS(LW)=CC(1)
   41 DO 71 L=2,LW
   42 DO 59 K=1,LW
   43 DO 45 I=1,LY
   44 LV=I+LY
   45 C(I,K)=A(I,LV)*B(LV,K)+C(I,K)
   46 DO 49 I=LX,LW
   47 LV=I-LY
   48 C(I,K)=A(I,LV)*B(LV,K)+C(I,K)
   49 C(I,K)=A(I,I)*B(I,K)+C(I,K)
   50 LR=LW-1
   51 DO 54 I=LX,LR
   52 LP=I-LU
   53 C(I,K)=A(I,LP)*B(LP,K)+C(I,K)
   54 C(I,K)=A(I,I+1)*B(I+1,K)+C(I,K)
   55 LO=2+LY
   56 DO 59 I=LO,LW
   57 LC=I-LX
   58 C(I,K)=A(I,LC)*B(LC,K)+C(I,K)
   59 C(I,K)=A(I,I-1)*B(I-1,K)+C(I,K)
   60 DO 64 I=1,LW
   61 SIGMA(L)=C(I,I)+SIGMA(L)
   62 DO 64 J=1,LW
   63 B(I,J)=C(I,J)
   64 C(I,J)=0.0D0
   65 P=L
   66 DO 71 J=1,L
   67 N=L-J+1
   68 M=LW+1-L
   69 CS(L)=CC(L-1)
   70 CC(L)=-(1.D0/P)*(CS(N)*SIGMA(J))+CC(L)
   71 SS(M)=CC(L)
   73 FORMAT(2X,4E20.7)
   74 RETURN
   75 END
```

```
      SUBROUTINE GAUSSR
      DIMENSION A(10,11),B(11),BB(10)
      COMMON/QUANIT/ZETA(11),C(10,11),ZL
      EQUIVALENCE (B,ZETA)
C       C MATRIX= MATRIX OF INFLUENCE COEFFICIENTS
C     READ STATEMENT AUGUMENTS  C MATRIX,READS B EQUIVALENT
    1 FORMAT(2X,5F15.8)
    2 FORMAT(2X,5E22.8)
    3 NUM=ZL
    4 LX=NUM+1
    5 READ(5,1)(C(1,LX),C(2,LX),C(3,LX),C(4,LX),C(5,LX),C(6,LX),C(7,LX)
    61,C(8,LX),C(9,LX),C(10,LX))
    7 WRITE(6,2)(C,NUM)
    8 LU=NUM-1
    9 DO 11  I=1,NUM
   10 DO 11  J=1,LX
   11 A(I,J)=C(I,J)
   12 DO 38  L=1,LU
   13 DO 14  II=L,NUM
   14 BB(II)=A(II,L)
   15 DO 21  KK=L,LU
   16 DO 21  JJ=KK,LU
   17 IF(ABS(BB(KK))-ABS(BB(JJ+1)))21,21,18
   18 Z=BB(KK)
   19 BB(KK)=BB(JJ+1)
   20 BB(JJ+1)=Z
   21 CONTINUE
   22 DO 25  JK=1,NUM
   23 IF(ABS(A(JK,L))-ABS(BB(NUM)))25,24,25
   24 LZ=JK
   25 CONTINUE
   26 DO 30  IH=1,LX
   27 Z=A(LZ,IH)
   28 A(LZ,IH)=A(L,IH)
   29 A(L,IH)=Z
   30 CONTINUE
   31 DO 38  I=1,LX
   32 B(I)=0.0
   33 K=NUM+2-I
   34 DO 35  J=L,NUM
   35 A(J,K)=A(J,K)/A(J,L)
   36 DO 37  J=L,LU
   37 A(J+1,K)=A(J+1,K)-A(L,K)
   38 CONTINUE
   39 DO 41  M=1,NUM
   40 I=LX-M
   41 B(I)=A(I,LX )/A(I,I)-A(I,2)*B(2)-A(I,3)*B(3)-A(I,4)*B(4)-A(I,5)*
  421B(5)-A(I,6)*B(6)-A(I,7)*B(7)-A(I,8)*B(8)-A(I,9)*B(9)-A(I,10)*B(10)
   43 WRITE(6,44)(B)
   44 FORMAT(2X,E20.8)
      RETURN
      END
```

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